Practice problems for the 430 final

Problem 1. *Recall that the soundness theorem says that if* $\Gamma \vdash \phi$*, then* $\Gamma \models \phi$*. Prove the soundness theorem, by assuming without proof that every logical axiom is valid (i.e. holds in every model).*

Problem 2. *Suppose that* Σ *is a set of sentences that has arbitrarily large finite models. Show that* Σ *has an infinite model.*

Problem 3. *Let* $\mathcal{L} = \{<\}$ *be a first order language where* $<$ *is a binary relation symbol.*

(1) *Show that the theory of infinite linear orders is axiomatizable.*
(2) *Show that the theory of infinite linear orders is not finitely axiomatizable.*

For the following problem you can make use of the formula $\phi_{code}(x)$ and that $\mathfrak{A} \models \phi_{code}[a]$ iff a codes a sequence. You can also use "$lh(a)$" to refer to the length of the sequence coded by a.

Problem 4. *Write down a* Σ_1 *formula* $\phi_{exp}(e, n, k)$*, such that* $\mathfrak{A} \models \phi_{exp}[e, n, k]$ *iff* $e^n = k$*. Then write down a* Π_1 *formula* $\phi'_{exp}(e, n, k)$ *equivalent to* $\phi_{exp}(e, n, k)$.

For the problems below, recall that any model of PA is an end extension of $\mathfrak{A}$, and as a corollary we get that if ϕ is Σ_1 and true in $\mathfrak{A}$, then ϕ is true in any model of PA, and so $PA \vdash \phi$. Recall also that we defined a Σ_1 formula $\phi_{prov-\theta}(a, b)$ such that, $\mathfrak{A} \models \phi_{prov-\theta}[a, b]$ iff $T_\theta \vdash \phi_a(b)$. Then setting e to be the Gödel number of $\neg\phi_{prov-\theta}(v, v)$, we defined

$$\sigma := \neg\phi_{prov-\theta}(e, e)$$

i.e. σ is exactly $\phi_e(e)$. Note that since $\phi_{prov-\theta}$ is Σ_1 and σ is defined by taking its negation, we have that σ is Π_1.

Problem 5. (1) *Show that* $\mathfrak{A} \models \sigma$ *iff* $T_\theta \not\vdash \sigma$*.*
(2) *Prove that* $T_\theta \not\vdash \sigma$ *(and so* $\mathfrak{A} \models \sigma$*). (Here you will use that* σ *is* Σ_1*)*
(3) *Prove that* σ *is not* Σ_1 *(and so not* Δ_1*).*

Problem 6. *Show that there is no formula* $\phi_{true}(x, y)$ *such that*

$$\mathfrak{A} \models \phi_{true}[a, b] \text{ iff } \mathfrak{A} \models \phi_a[b].$$

Hint: suppose for contradiction that such a formula exists. Define a sentence σ' *is a similar fashion as* σ *from above. I.e. informally,* σ' *will be the sentence "I am not true".*

Also make sure you know:

- The statements of Soundness, Compactness, Completeness theorems.
- How to show Compactness assuming Completeness; how to prove Soundness.
- The statements and proofs of the First and Second Incompleteness theorems.

- The definition of $\Delta_0, \Sigma_1, \Pi_1, \Delta_1$ formulas.
- How to show that there exists a countable nonstandard model of PA, i.e. a model $\mathfrak{B}$ which is not isomorphic to $\mathfrak{A}$. Here the proof uses Compactness, to construct a model with an "infinite" element, i.e. an element $b \in \mathfrak{B}$ such that for all $n \in \mathbb{N}$, $S^n_{\mathfrak{B}}(0_{\mathfrak{B}}) <_{\mathfrak{B}} b$.
- How to prove that every model of PA is an end extension of $\mathfrak{A}$.
- As a corollary to the above: that if ϕ is Σ_1 and $\mathfrak{A} \models \phi$, then any model of PA $\mathfrak{B} \models \phi$. Note that this fact is a key ingredient when showing the First Incompleteness theorem.